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DISCOVERING HIDDEN TOTALLY LEONTIEF SUBSTITUTION SYSTEMS.(U)

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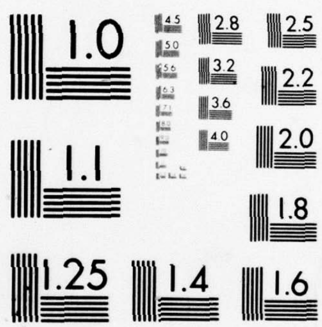

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BY

GEORGE B. DANTZIG and ARTHUR F. VEINOTT, JR.

TECHNICAL REPORT NO. 32
DECEMBER 20, 1977

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DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
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by 9 Technical rept.,

10 George B. Dantzig [redacted] Arthur F. Veinott, Jr.

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Abstract

A constructive procedure is given for determining the existence of and evaluating (when it does exist) a nonsingular matrix that transforms a system of linear equations in nonnegative variables into a totally Leontief substitution system. The computational effort involved is about that required to optimize the given m-row linear system with m+1 different linear objective functions.

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The system of m linear equations in n nonnegative variables

$$(1) \qquad Ax = b \ , \quad x \geq 0$$

is called a Leontief substitution system [2] if (i) each column A_j of A has at most one positive element, (ii) $b \gg 0$ and (iii) the set of solutions to (1) is nonempty. If also that set is bounded, (1) is called a totally Leontief substitution system [6]. In either case, it is known that A has rank m. Such systems are discussed in [1] - [8].

Saigal [8], [7] calls (1) a hidden totally Leontief substitution system if there exists a nonsingular matrix Π such that

$$(2) \qquad (\Pi A)x = \Pi b \ , \quad x \geq 0$$

is a totally Leontief substitution system. The purpose of this paper is to give a constructive method for determining whether or not (1) has this property, and if so, to find Π .

Substitution Classes. Associated with any feasible $m \times m$ basis $B = (B_i)$ for (1) is, for $i = 1,\ldots,m$, the set S_i of column indices j such that A_j , if substituted for B_i , forms a feasible basis. Each substitution class S_i is nonempty since it includes a j with

$A_j = B_i$. In general the S_i depend on B and b and can be overlapping when there are degenerate basic feasible solutions. For the totally Leontief substitution case, however, the S_i are independent of the choice of B and $b >> 0$; indeed, S_i consists of all j such that A_j has a positive element in the same row as B_i . Also the S_i partition the column indices $1,\ldots,n$. Thus every submatrix B consisting of m columns of A with a positive element in each row forms a (nondegenerate) feasible basis, and conversely.

<u>The Algorithm</u>.

<u>Step 1</u>. Find a feasible $m \times m$ basis B and determine substitution classes $S_1,\ldots,S_m$ with respect to B , b . Terminate if there is no feasible basis or if the substitution classes do not partition the column indices of A . Otherwise go to Step 2.

<u>Step 2</u>. Solve the linear program of maximizing $z = \Sigma x_j$ subject to (1). Terminate if z is unbounded above. Otherwise go to Step 3.

<u>Step 3</u>. For each $i = 1,\ldots,m$, determine the i^{th} row of Π as any vector π_i such that

$$\begin{aligned} &\pi_i b > 0 \\ (3) \qquad & \\ &\pi_i A_j \leq 0 \quad \text{for all} \quad j \notin S_i \ . \end{aligned}$$

Terminate if for any $i = 1,\ldots,m$ the system (3) is infeasible. Otherwise terminate with Π .

Theorem. If the algorithm terminates with Π , then Π is nonsingular and (2) is a totally Leontief substitution system. Otherwise (1) is not a hidden totally Leontief substitution system.

Proof. If (1) is a hidden totally Leontief substitution system, then there is a feasible basis, the associated substitution classes partition the column indices of A and z in Step 2 is bounded above, because these properties are invariant under nonsingular transformations Π . Also there is a nonsingular matrix Π whose i^{th} row π_i satisfies (3) for each i . Thus if the algorithm terminates without obtaining Π , then (1) is not a hidden totally Leontief substitution system.

If the algorithm does terminate with Π, then ΠA has at most one positive element in each column (from Steps 1 and 3), $\Pi b >> 0$ (from Step 3) and (2) has a solution (from Step 1), so (2) is a Leontief substitution system. Hence ΠA has rank m, implying Π is nonsingular and so (1) and (2) have the same solution set. Thus the boundedness of the solution set of (1) implies that is so of (2), so (2) is a totally Leontief substitution system.

Computational Remarks. The computational effort required to execute the algorithm is about that required to solve the linear program of minimizing cx subject to (1) with $m+1$ different objective-function vectors $c = (c_j)$. To determine the substitution classes in Step 1 requires computing $b' = (b'_k) = B^{-1}b$ and $A'_j = (A'_{jk}) = B^{-1}A_j$ for each j . Then the substitution classes partition the column indices of A if and only if for each j there is a unique $k = k(j)$ that minimizes b'_k/A'_{jk} subject to $A'_{jk} > 0$. In that event $j \in S_{k(j)}$ for each j . Step 2

involves solving the linear program with $c_j = -1$ for all j. Finally, Step 3 necessitates solving m linear programs. The i^{th}, $1 \leq i \leq m$, of these has $c_j = 1$ for all $j \in S_i$ and $c_j = 0$ otherwise. If optimal simplex multipliers π_i exist therefor, they satisfy (3). If no such multipliers exist, (3) is infeasible. Incidentally, Step 3 can be streamlined somewhat by modifying the i^{th} linear program so that all but an (arbitrary) one of the variables x_j with $j \in S_i$ is omitted.

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